# Inclass 12: Significance Test

## [SCS4049] Machine Learning and Data Science

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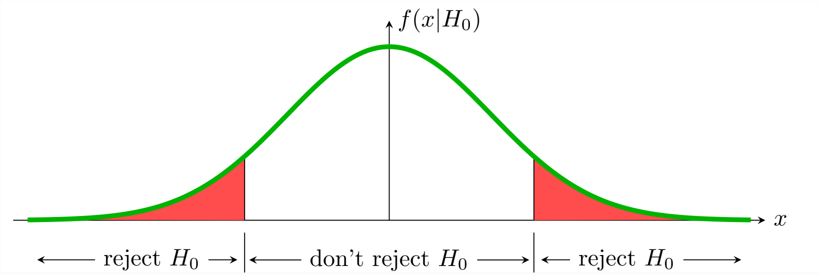
# Significance Test

# Hypothesis test

Hypothesis tests, also called significance tests: to help you learn whether random change might be responsible for an observed effect

- **Null hypothesis**: the hypothesis that change is to blame
- **Alternative hypothesis**: Counterpoint to the null (what you hope to prove)
- **One-way test**: Hypothesis test that counts change results only in one direction
- **Two-way test**: Hypothesis test that counts change results in two directions

# Significance test




- $\mathbf{x}$ : test statistic
- $f(\mathbf{x} \mid \mathcal{H})$ : likelihood of null hypothesis
- Rejection region: a portion of the x-axis where null hypothesis will be rejected
- Significance: probability over the rejection region (red area)

# Significance test

For a significance level $\alpha$, or so called $\alpha$-level significance test,

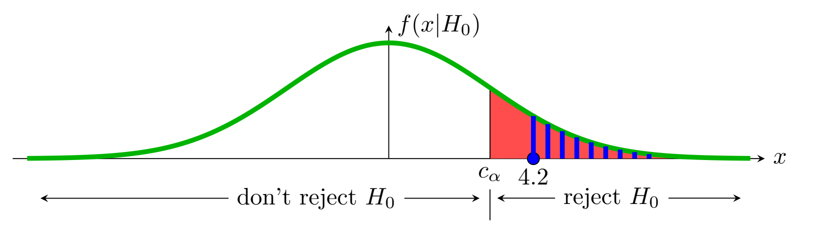
$$\alpha = P(\mathbf{x} \text{ in rejection region} \mid \mathcal{H}_0) \tag{1}$$

The p-value is a tool to check if the test statistic is in the rejection region. It is also a measure of the evidence for rejecting $\mathcal{H}_0$.

$$\text{p-value} = P(\text{data at least as extreme as } x \mid \mathcal{H}_0) \tag{2}$$

# Significance test: example

Suppose we have the right-sided rejection region shown below. Also suppose we see data with test statistic $x = 4.2$. Should we reject $\mathcal{H}_0$?



# Critical value

- The boundary of the rejection region are called *critical values*
- Critical values are labeled by the probability to their right
- Example: for a standard normal $c_{0.025} = 1.96$ and $c_{0.975} = -1.96$

# Likelihood Ratio Test

# Likelihood ratio

We often want to test in situations where the adopted probability model involves several unknown parameters. Thus, we may denote an element of the parameter space by

$$\boldsymbol{\theta} = (\theta_1, \theta_2, ..., \theta_k). \tag{3}$$

We use the *likelihood ratio*, $\lambda(\mathbf{x})$, defined as

$$\lambda(\mathbf{x}) = \frac{\sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}_0} L(\boldsymbol{\theta}; \mathbf{x})}{\sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} L(\boldsymbol{\theta}; \mathbf{x})}. \tag{4}$$

The informal argument for this is as follows.

# Likelihood ratio

For a realisation $\mathbf{x}$, determine its best chance of occurrence under $\mathcal{H}_0$ and also its best chance overall. The ratio of these two chances can never exceed unity, but, if small, would constitute evidence for rejection of the null hypothesis.

A *likelihood ratio test* for testing $\mathcal{H}_0 : \boldsymbol{\theta} \in \boldsymbol{\Theta}_0$ against $\mathcal{H}_1 : \boldsymbol{\theta} \in \boldsymbol{\Theta}_1$ is a test with critical region of the form

$$C_1 = \{\mathbf{x}:\lambda(\mathbf{x}) \leq k\} \tag{5}$$

where $k$ is a real number between 0 and 1.

# Likelihood ratio: example

**Example** Exponential distribution

$$\mathcal{H}_0 : \theta = \theta_0 \qquad \mathcal{H}_1 : \theta > \theta_0 \tag{6}$$

Then,

$$\mathbf{\Theta}_0 = \{\theta_0\} \qquad \mathbf{\Theta}_1 = [\theta_0, \infty). \tag{7}$$

The likelihood function is

$$L(\theta; \mathbf{x}) = \prod_{n=1}^{N} f(x_n; \theta) = \theta^N e^{-\theta \sum x_n}. \tag{8}$$

# Likelihood ratio: example

The numerator of the likelihood ratio is

$$L(\theta_0; \mathbf{x}) = \theta_0^N e^{-N\theta_0 \bar{x}}. \tag{9}$$

We need to find the supremum as $\theta$ ranges over the interval $[\theta_0, \infty)$. Now

$$\log L(\theta; \mathbf{x}) = N \log \theta - N\theta\bar{x} \tag{10}$$

so that

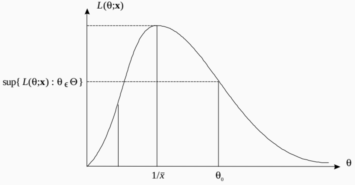
$$\frac{\partial}{\partial\theta} \log L(\theta; \mathbf{x}) = \frac{N}{\theta} - N\theta\bar{x} \tag{11}$$

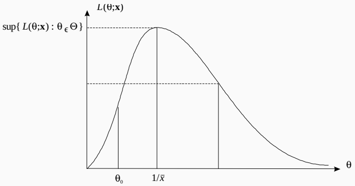
which is zero only when $\theta = 1/\bar{x}$. Since $L(\theta; \mathbf{x})$ is an increasing function for $\theta < 1/\bar{x}$ and decreasing for $\theta > 1/\bar{x}$,

$$\sup_{\theta\in\Theta} L(\theta; \mathbf{x}) = \begin{cases} \bar{x}^{-N} e^{-N} & \text{if } 1/\bar{x} \geq \theta_0 \\ \theta_0^N e^{-N\theta_0\bar{x}} & \text{if } 1/\bar{x} < \theta_0. \end{cases} \tag{12}$$

# Likelihood ratio: example

$$\lambda(\mathbf{x}) = \begin{cases} \frac{\theta_0^N e^{-N\theta_0\bar{x}}}{\bar{x}^{-N}e^{-N}} & \text{if } 1/\bar{x} \geq \theta_0 \\ 1 & \text{if } 1/\bar{x} < \theta_0 \end{cases} = \begin{cases} \theta_0^N \bar{x}^N e^{-N\theta_0\bar{x}} e^N & \text{if } 1/\bar{x} \geq \theta_0 \\ 1 & \text{if } 1/\bar{x} < \theta_0 \end{cases} \tag{13}$$



# Likelihood ratio: example

Since $\frac{d}{d\bar{x}}(\cdot)$ is positive for values of $\bar{x}$ between 0 and $1/\theta_0$ where $\theta_0 > 0$, it follows that $\lambda(\mathbf{x})$ is a non-decreasing function of $\bar{x}$. Therefore the critical region of the likelihood ratio test is of the form

$$C_1 = \left\{ \mathbf{x} \colon \sum_{n=1}^{N} x_n \leq c \right\}. \tag{14}$$