Preclass 09: Singular Value Decomposition

[SCS4049] Machine Learning and Data Science

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Singular value decomposition

Singular value decomposition (SVD) of a given matrix A

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T \tag{1}$$

$$= \begin{bmatrix} | & | & & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_r \\ | & | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_k \end{bmatrix} \begin{bmatrix} — & \mathbf{v}_1^T & — \\ — & \mathbf{v}_2^T & — \\ & \vdots & \\ — & \mathbf{v}_r^T & — \end{bmatrix} \tag{2}$$

where

- $\mathbf{A} \in \Re^{m \times n}, \ \mathrm{rank}(\mathbf{A}) = r$
- $\mathbf{U} \in \Re^{m \times r}, \ \mathbf{U}^T\mathbf{U} = \mathbf{I}$
- $\mathbf{V} \in \Re^{n \times r}, \ \mathbf{V}^T\mathbf{V} = \mathbf{I}$
- $\mathbf{\Sigma} = \mathrm{diag}(\sigma_1, ..., \sigma_r)$ where $\sigma_1 \geq \cdots \geq \sigma_r > 0$

Singular value decomposition

with $\mathbf{U} = \begin{bmatrix} | & | & & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_r \\ | & | & & | \end{bmatrix}$ and $\mathbf{V} = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_r \\ | & | & & | \end{bmatrix}$,

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T = \sum_{k=1}^{r} \sigma_k \mathbf{u}_k \mathbf{v}_k^T \tag{3}$$

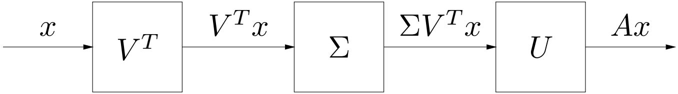
where

- σ_i are the nonzero *singular values* of $\mathbf{A}$
- $\mathbf{v}_i$ are the *right* or *input singular vectors* of $\mathbf{A}$
- $\mathbf{u}_i$ are the *left* or *output singular vectors* of $\mathbf{A}$

Interpretations

SVD:

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T = \sum_{k=1}^{r} \sigma_k \mathbf{u}_k \mathbf{v}_k^T \tag{4}$$



Linear mapping $\mathbf{y} = \mathbf{A}\mathbf{x}$ can be decomposed as

- compute coefficients of $\mathbf{x}$ along input directions $\mathbf{v}_1, \ldots, \mathbf{v}_r$
- scale coefficients by σ_i
- reconstitute along output directions $\mathbf{u}_1, \ldots, \mathbf{u}_r$

difference with eigenvalue decomposition for symmetric $\mathbf{A}$: input and output directions are *different*

Geometric interpretation